Toggling Antichains of Posets

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Outline

- Background: toggles and rowmotion on order ideals and antichains of posets.
- Focus on the antichain toggle group and construct an explicit isomorphism with the order ideal toggle group.
- Generalize to Stanley's order polytope and chain polytope of posets.

The toggle group of order ideals

Let P be a poset. Let $\mathcal{A}(P)$, $J(P)$, $F(P)$ denote the sets of antichains, order ideals, and order filters of P respectively.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $e \in P$. Then the **order ideal toggle** corresponding to e is the map $t_e : J(P) \to J(P)$ defined by

$$t_e(I) = \begin{cases} I \cup \{e\} & \text{if } e \notin I \text{ and } I \cup \{e\} \in J(P), \\ I \setminus \{e\} & \text{if } e \in I \text{ and } I \setminus \{e\} \in J(P), \\ I & \text{otherwise.} \end{cases}$$

The toggle group of order ideals

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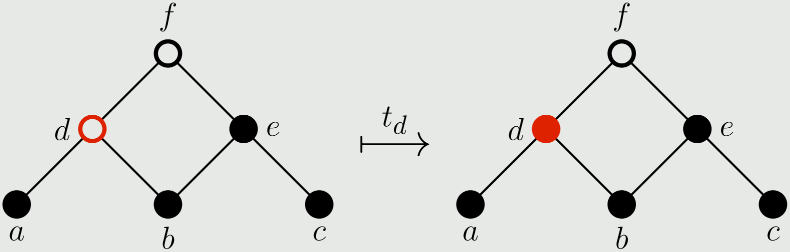
Example



Since d is not in the original order ideal, and adding d results in an order ideal, we add d in.

The toggle group of order ideals

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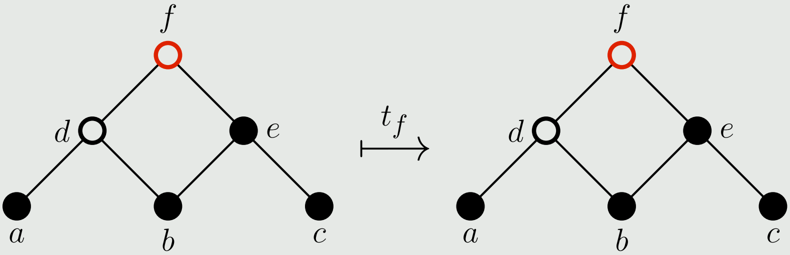

Now f is not in the original order ideal, but adding f does **not** result in an order ideal. So t_f does nothing.

The toggle group of order ideals

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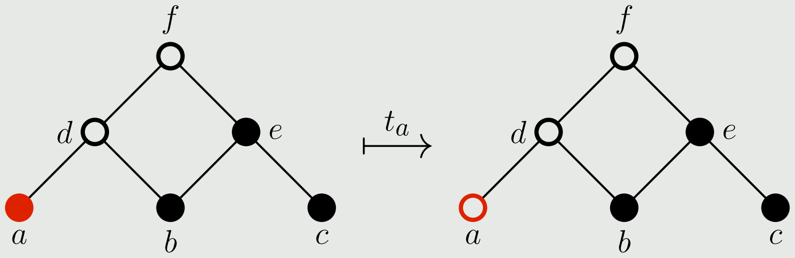
Example



Since a is in the order ideal, and removing a still results in an order ideal, we remove a.

The toggle group of order ideals

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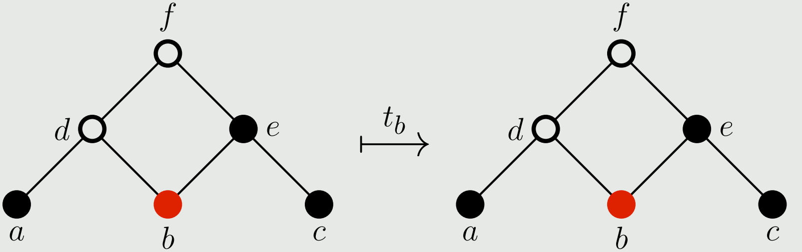
Example



Since b is in the order ideal, and removing b does **not** result in an order ideal, t_b does nothing.

The toggle group of order ideals

Definition

The **toggle group** of $J(P)$, denoted $\mathrm{Tog}_J(P)$, is the subgroup of $\mathfrak{S}_{J(P)}$ generated by all toggles $\{t_e \mid e \in P\}$.

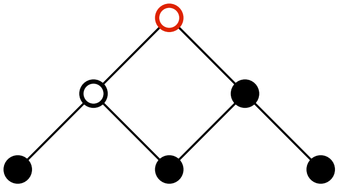
Theorem (Cameron and Fon-Der-Flaass 1995)

For a finite connected poset P, $\mathrm{Tog}_J(P)$ is either the symmetric group $\mathfrak{S}_{J(P)}$ or alternating group $\mathfrak{A}_{J(P)}$ on $J(P)$.

Order ideal rowmotion

One particular element of the toggle group $\mathrm{Tog}_J(P)$ is called **order ideal rowmotion** and denoted Row_J.

Let $(x_1, x_2, \ldots, x_n)$ be any linear extension of a finite poset P. Then $\mathrm{Row}_J = t_{x_1} t_{x_2} \cdots t_{x_n}$.



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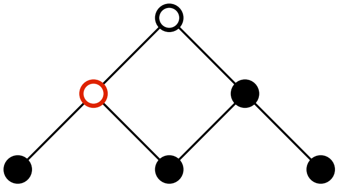

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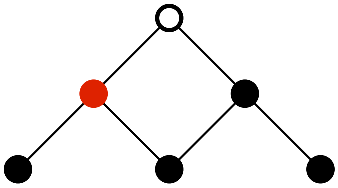

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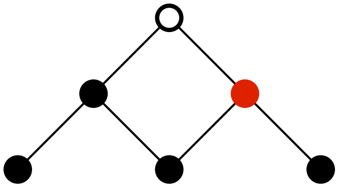

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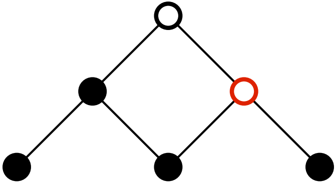

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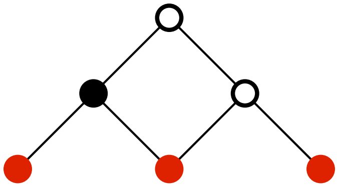

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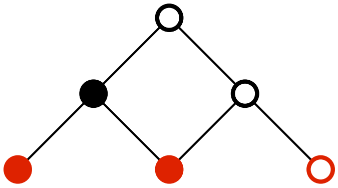

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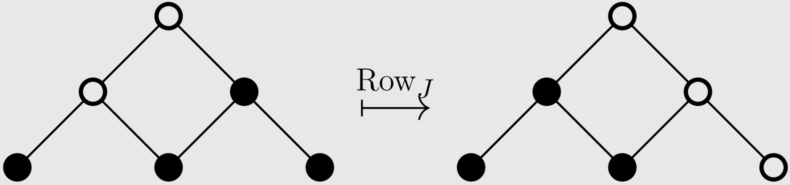


Rowmotion

For some specific families of posets (e.g. root posets, zigzag posets, products of chains), various phenomena have been discovered for rowmotion including

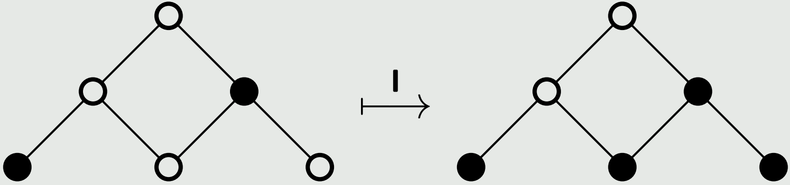
- the order of the map being easy to describe in general
- cyclic sieving
- homomesy
- resonance
- equivariant bijections

Another way to describe rowmotion

There are natural bijections between $\mathcal{A}(P)$, $J(P)$, and $F(P)$.

- Complementation is a bijection between $J(P)$ and $F(P)$.
- An antichain A generates an order ideal $\mathbf{I}(A) := \{x \in P \mid x \leq y \text{ for some } y \in A\}$ whose set of maximal elements is A.
- An antichain A generates an order filter $\mathbf{F}(A) := \{x \in P \mid x \geq y \text{ for some } y \in A\}$ whose set of minimal elements is A.

Example



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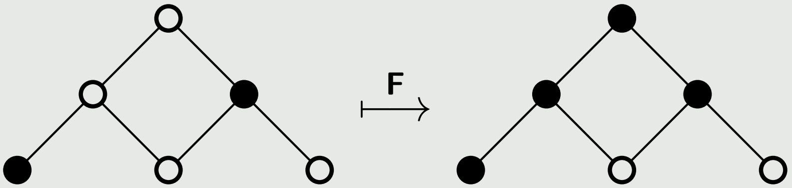

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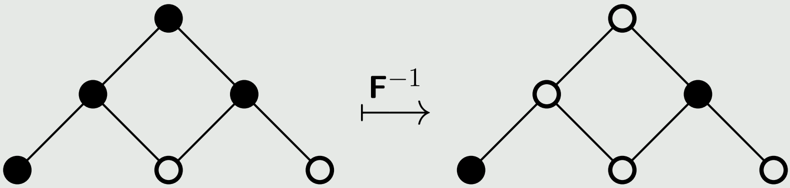
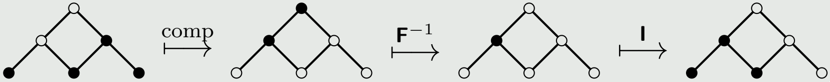


Another way to describe rowmotion

$$\mathrm{Row}_J \quad : \quad J(P) \xrightarrow{\text{comp}} F(P) \xrightarrow{\mathbf{F}^{-1}} \mathcal{A}(P) \xrightarrow{\mathbf{I}} J(P)$$

- complement
- take minimal elements
- generate order ideal

Example (Row_J)

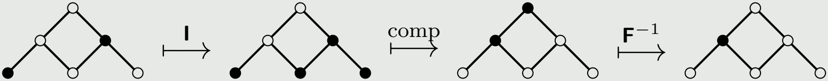


Antichain rowmotion

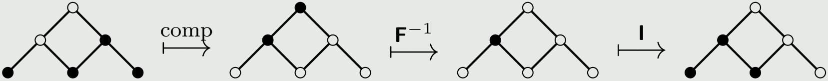
$$\mathrm{Row}_{\mathcal{A}} \quad : \quad \mathcal{A}(P) \xrightarrow{\mathbf{I}} J(P) \xrightarrow{\text{comp}} F(P) \xrightarrow{\mathbf{F}^{-1}} \mathcal{A}(P)$$

$$\mathrm{Row}_{J} \quad : \quad J(P) \xrightarrow{\text{comp}} F(P) \xrightarrow{\mathbf{F}^{-1}} \mathcal{A}(P) \xrightarrow{\mathbf{I}} J(P)$$

Example ($\mathrm{Row}_{\mathcal{A}}$)



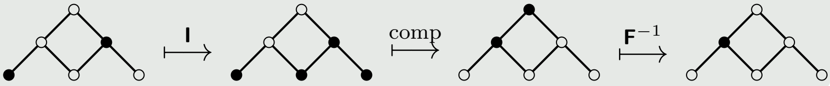
Example (Row_{J})



Antichain rowmotion

We call $\mathrm{Row}_{\mathcal{A}}$ **antichain rowmotion**.

Example ($\mathrm{Row}_{\mathcal{A}}$)



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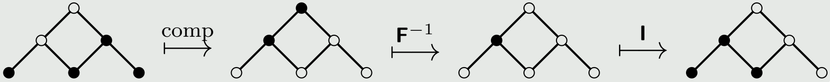

$$\begin{array}{ccc} \mathcal{A}(P) & \xrightarrow{\mathrm{Row}_{\mathcal{A}}} & \mathcal{A}(P) \\ \mathbf{I}\downarrow & & \downarrow\mathbf{I} \\ J(P) & \xrightarrow[\mathrm{Row}_{J}]{} & J(P) \end{array}$$

The toggle group of antichains

Striker has generalized the notion of toggles relative to any set of “allowed” subsets, not necessarily order ideals.

Definition

Let $e \in P$. Then the **antichain toggle** corresponding to e is the map $\tau_e : \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\tau_e(A) = \begin{cases} A \cup \{e\} & \text{if } e \notin A \text{ and } A \cup \{e\} \in \mathcal{A}(P), \\ A \setminus \{e\} & \text{if } e \in A, \\ A & \text{otherwise.} \end{cases}$$

Let $\mathrm{Tog}_{\mathcal{A}}(P)$ denote the **toggle group** of $\mathcal{A}(P)$ generated by the toggles $\{\tau_e \mid e \in P\}$.

The toggle group of antichains

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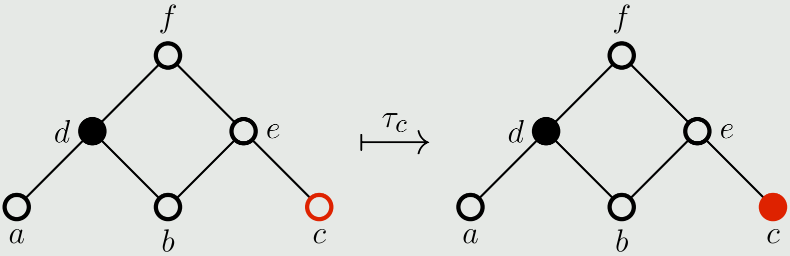

Adding c results in an antichain, so τ_c adds c in.

The toggle group of antichains

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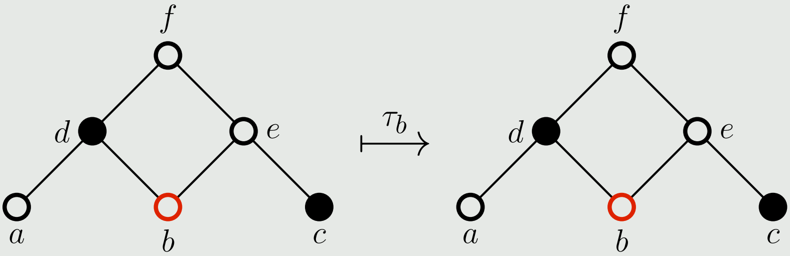

Adding b does not result in an antichain, so τ_b does nothing.

The toggle group of antichains

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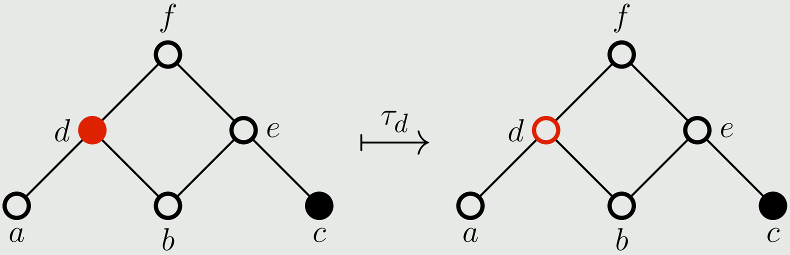
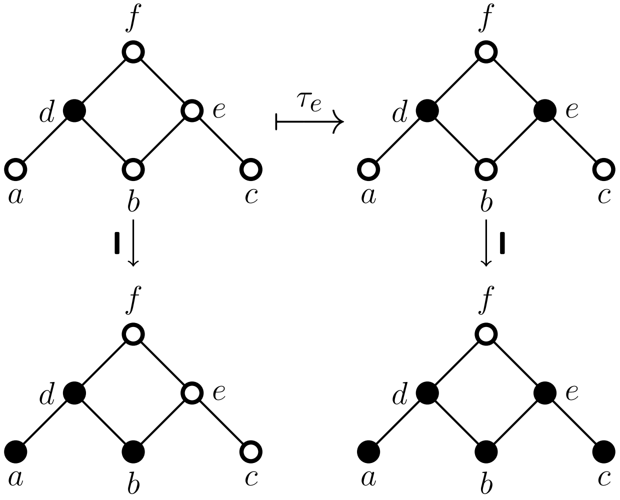


Notice that (unlike order ideals) we can **always** remove an element out of an antichain and still get an antichain.

Antichain and order ideal toggles are different



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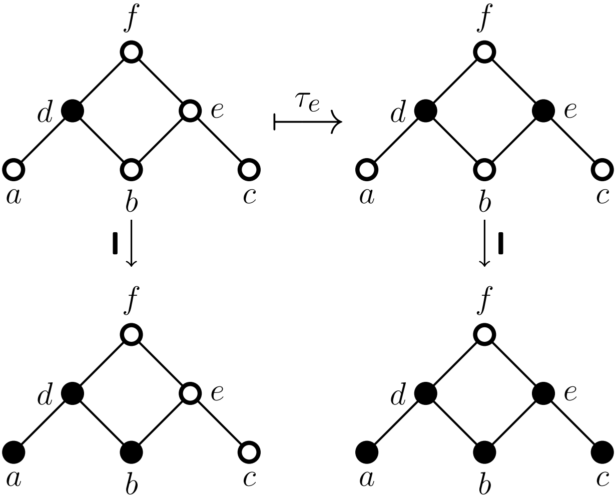

Note that we use t_e to refer to the order ideal toggles, and we use τ_e to refer to the antichain toggles.

The toggle groups

Theorem (Cameron and Fon-Der-Flaass 1995)

For a finite connected poset P, $\mathrm{Tog}_J(P)$ is either the symmetric group $\mathfrak{S}_{J(P)}$ or alternating group $\mathfrak{A}_{J(P)}$ on $J(P)$.

Theorem (Striker)

For a finite connected poset P, $\mathrm{Tog}_{\mathcal{A}}(P)$ is either the symmetric group $\mathfrak{S}_{\mathcal{A}(P)}$ or alternating group $\mathfrak{A}_{\mathcal{A}(P)}$ on $\mathcal{A}(P)$.

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Now we will describe an explicit isomorphism between $\mathrm{Tog}_J(P)$ and $\mathrm{Tog}_{\mathcal{A}}(P)$.

How order ideal toggles act on antichains

Definition

For $e \in P$, let $\{e_1, \ldots, e_k\}$ be the (possibly empty) set of elements that e covers. Define $t_e^* \in \mathrm{Tog}_{\mathcal{A}}(P)$ as

$$t_e^* := \tau_{e_1}\tau_{e_2}\cdots\tau_{e_k}\tau_e\tau_{e_1}\tau_{e_2}\cdots\tau_{e_k}$$

In particular, if e is a minimal element of P, then $t_e^* = \tau_e$.

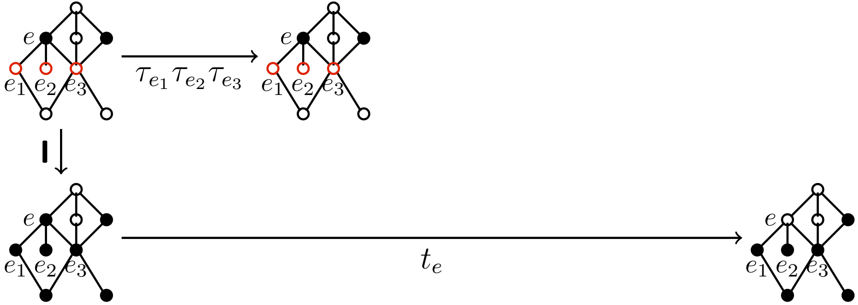
Theorem (J.)

The following diagram commutes.

$$\begin{array}{ccc}
\mathcal{A}(P) & \xrightarrow{\;t_e^*\;} & \mathcal{A}(P) \\
\mathbf{I}\big\downarrow & & \big\downarrow\mathbf{I} \\
J(P) & \xrightarrow[\;t_e\;]{} & J(P)
\end{array}$$

How order ideal toggles act on antichains

$$t_e^* = \tau_{e_1}\tau_{e_2}\tau_{e_3}\tau_e\tau_{e_1}\tau_{e_2}\tau_{e_3}$$



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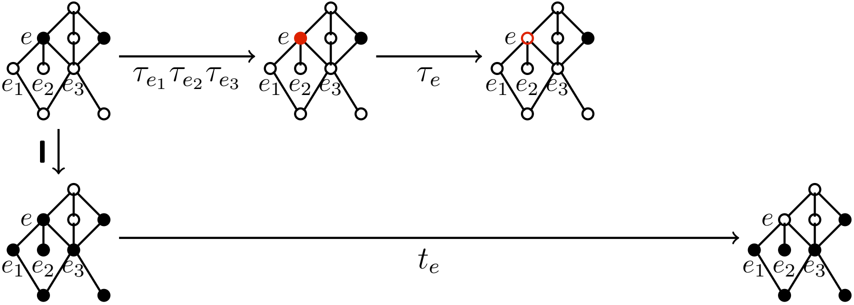

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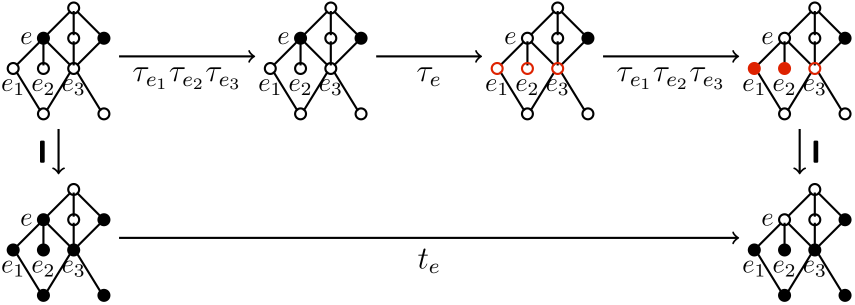


How antichain toggles act on order ideals

Definition

- $\eta_e := t_{x_1} t_{x_2} \cdots t_{x_k}$ where $(x_1, x_2, \ldots, x_k)$ is a linear extension of the subposet $\{x < e\}$ of P.
- If e is minimal in P, then η_e is the identity.
- $\tau_e^* := \eta_e t_e \eta_e^{-1}$

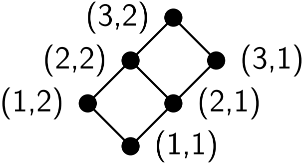
Theorem (J.)

The following diagram commutes.

$$\begin{array}{ccc} \mathcal{A}(P) & \xrightarrow{\ \tau_e\ } & \mathcal{A}(P) \\ \mathbf{I} \big\downarrow & & \big\downarrow \mathbf{I} \\ J(P) & \xrightarrow[\ \tau_e^*\ ]{} & J(P) \end{array}$$

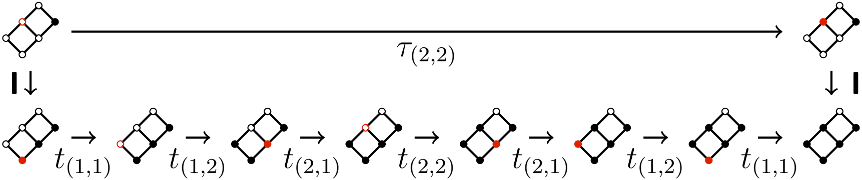
How antichain toggles act on order ideals

$P =$



$$\eta_{(2,2)} = t_{(1,1)}t_{(1,2)}t_{(2,1)}$$

$$\tau^*_{(2,2)} = \eta_{(2,2)}t_{(2,2)}\eta^{-1}_{(2,2)} = t_{(1,1)}t_{(1,2)}t_{(2,1)}t_{(2,2)}t_{(2,1)}t_{(1,2)}t_{(1,1)}$$



Isomorphism between $\mathrm{Tog}_{\mathcal{A}}(P)$ and $\mathrm{Tog}_J(P)$

Corollary (J.)

There is an isomorphism from $\mathrm{Tog}_{\mathcal{A}}(P)$ *to* $\mathrm{Tog}_J(P)$ *given by* $\tau_e \mapsto \tau_e^*$, *with inverse given by* $t_e \mapsto t_e^*$.

Isomorphism between $\mathrm{Tog}_{\mathcal{A}}(P)$ and $\mathrm{Tog}_{J}(P)$

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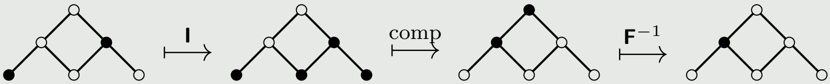
Theorem (Striker, J.)

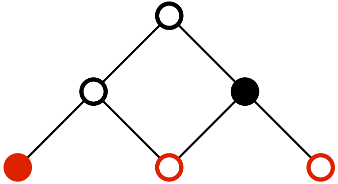
Let $(x_1, x_2, \ldots, x_n)$ *be any linear extension of a finite poset* P*. Then* $\mathrm{Row}_{\mathcal{A}} = \tau_{x_n} \cdots \tau_{x_2} \tau_{x_1}$*.*

Note that antichain rowmotion is a product of antichain toggles, just as order ideal rowmotion is a product of order ideal toggles, but in the *opposite* order.

Antichain rowmotion as a product of toggles

Example ($\mathrm{Row}_{\mathcal{A}}$)





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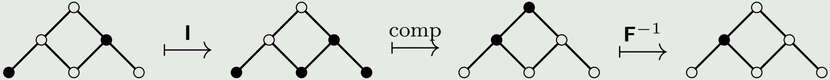

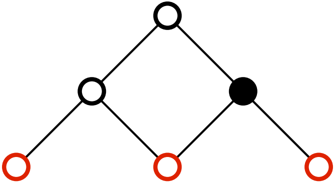

Antichain rowmotion as a product of toggles

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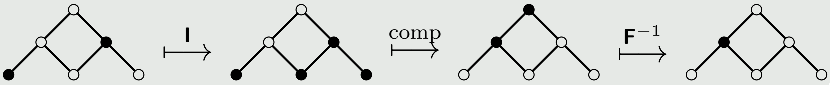

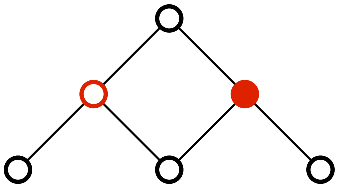

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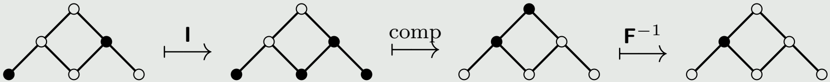

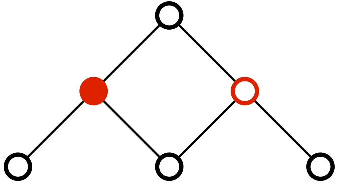

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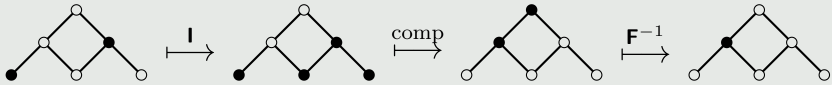
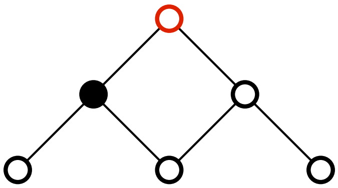
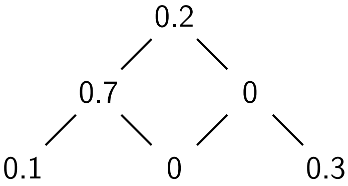




Poset polytopes

Stanley (1986) defined some polytopes associated with posets.

- $\mathcal{C}(P)$ is the **chain polytope** of P, the set of $f \in [0,1]^P$ such that $\sum_{i=1}^{n} f(x_i) \leq 1$ for all chains $x_1 < x_2 < \cdots < x_n$.
- $OR(P)$ is the **order-reversing polytope** of P, the set of all order-reversing labelings $f \in [0,1]^P$.
- $OP(P)$ is the **order-preserving polytope** of P, the set of all order-preserving labelings $f \in [0,1]^P$.



$\in \mathcal{C}(P)$

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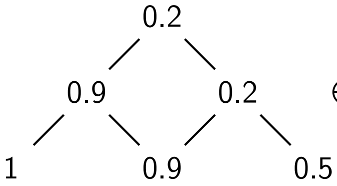

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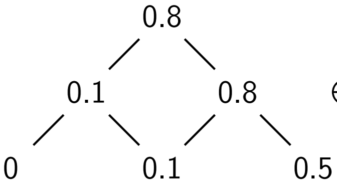
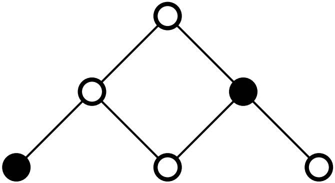


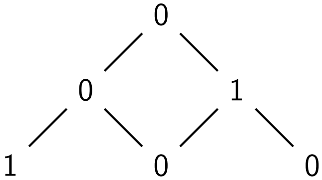
$\in OP(P)$

Poset polytopes

We can associate an **indicator function** to any subset of P.



$\longleftrightarrow$



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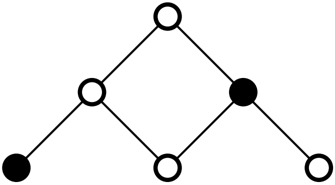

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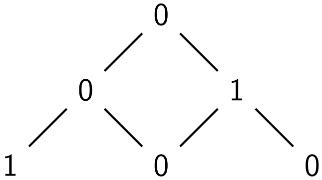

$\mathcal{A}(P), J(P), F(P)$ are precisely the vertices of these polytopes in which every element of P is labeled with 0 or 1.

$$\begin{aligned} \mathcal{A}(P) &= \mathcal{C}(P) \cap \{0,1\}^P \\ J(P) &= \mathcal{OR}(P) \cap \{0,1\}^P \\ F(P) &= \mathcal{OP}(P) \cap \{0,1\}^P \end{aligned}$$

The poset $\hat{P}$

The goal now is to extend the definitions of toggles from the combinatorial sets $\mathcal{A}(P)$, $J(P)$ to polytopes $\mathcal{C}(P)$, $OR(P)$.

The poset $\hat{P}$

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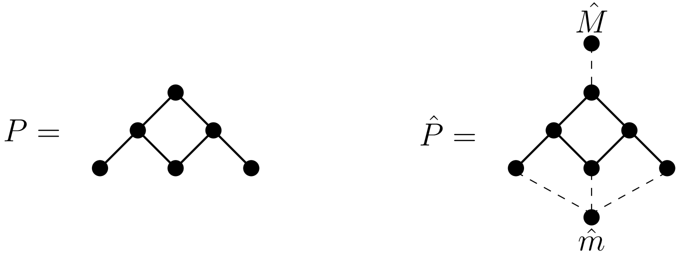
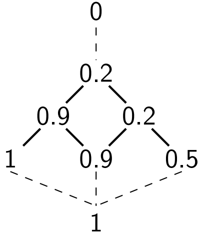
For any poset P, we will refer to a new poset $\hat{P}$ by adjoining a new minimal element $\hat{m}$ and new maximal element $\hat{M}$.

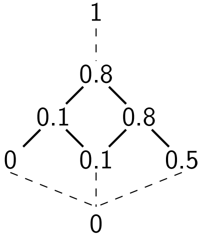


Extending labelings from P to $\hat{P}$

For $f \in OR(P)$, we extend to $\hat{P}$ by setting $f(\hat{m}) = 1$ and $f\left(\hat{M}\right) = 0$.



For $f \in OP(P)$, we extend to $\hat{P}$ by setting $f(\hat{m}) = 0$ and $f\left(\hat{M}\right) = 1$.



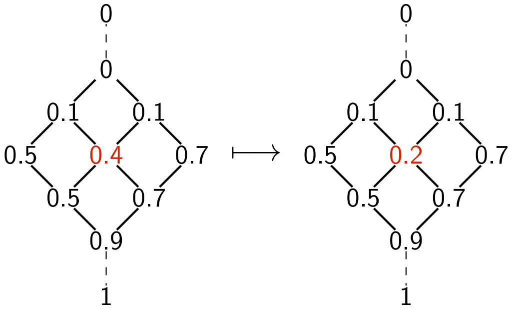
We need **not** extend elements of the chain polytope to $\hat{P}$.

Toggles on $OR(P)$

Definition (Einstein and Propp)

Let $e \in P$. Then we define $t_e : OR(P) \to OR(P)$ as follows.
Let $L = \max_{y \gtrdot e} f(y)$ and $R = \min_{y \lessdot e} f(y)$ in $\hat{P}$.

$$(t_e(f))(x) = \begin{cases} f(x) & \text{if } x \neq e \\ L + R - f(e) & \text{if } x = e \end{cases}$$



$0.1 + 0.5 - 0.4 = 0.2$

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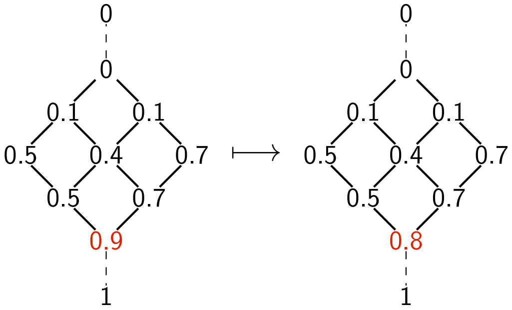
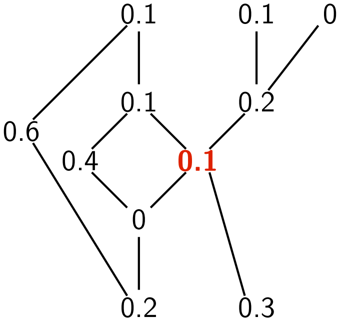


$$0.7 + 1 - 0.9 = 0.8$$

Toggles on the chain polytope $\mathcal{C}(P)$

To define $\tau_e : \mathcal{C}(P) \to \mathcal{C}(P)$, given $g \in \mathcal{C}(P)$ and $e \in P$, $\tau_e(g)$ can only differ from g at the value of e.

$$(\tau_e(g))(e) = 1 - \max \left\{ \sum_{i=1}^{k} g(y_i) \,\middle|\, \begin{array}{c} (y_1, \ldots, y_k) \text{ is a chain} \\ \text{in } P \text{ that contains } e \end{array} \right\}$$



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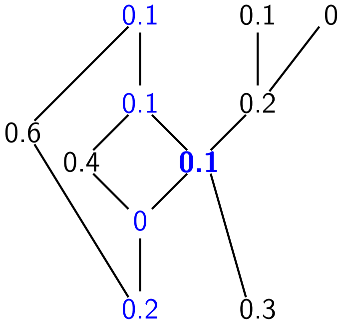

$$0.2 + 0 + 0.1 + 0.1 + 0.1 = 0.5$$

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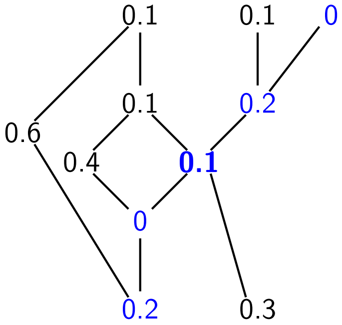

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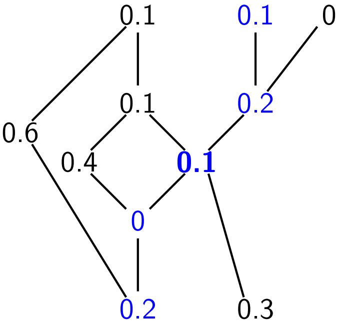

$$0.2 + 0 + 0.1 + 0.2 + 0.1 = 0.6$$

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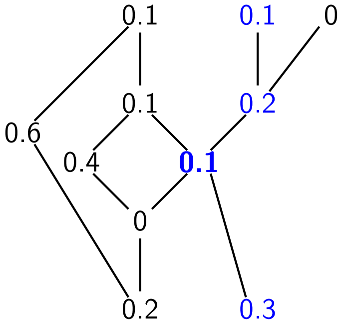

$$0.3 + 0.1 + 0.2 + 0.1 = 0.7$$

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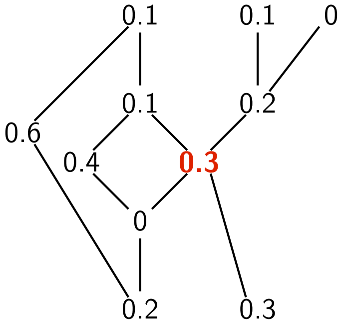

0.7 is max and $1 - 0.7 = 0.3$

Piecewise-linear toggling vs. combinatorial toggling

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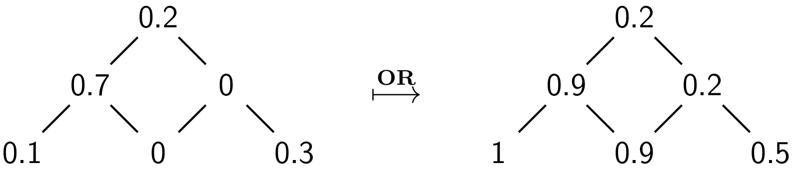
- However, for certain "nice" posets (like products of two chains), various properties of combinatorial rowmotion (surprisingly) extend to piecewise-linear (and furthermore birational) rowmotion (Einstein-Propp, Grinberg-Roby).
- The isomorphism from earlier between $\mathrm{Tog}_{\mathcal{A}}(P)$ and $\mathrm{Tog}_{J}(P)$ lifts to a piecewise-linear isomorphism between $\mathrm{Tog}_{\mathcal{C}}(P)$ and $\mathrm{Tog}_{OR}(P)$.

Bijections between $\mathcal{C}(P)$, $OR(P)$, and $OP(P)$

Stanley's "transfer map" gives a natural extension of

- the bijection **I** : $\mathcal{A}(P) \to J(P)$ to $\mathbf{OR} : \mathcal{A}(P) \to OR(P)$,
- the bijection **F**: $\mathcal{A}(P) \to F(P)$ to $\mathbf{OP} : \mathcal{A}(P) \to OP(P)$.



Relation between toggles on $\mathcal{C}(P)$ and $OR(P)$

Definition

- $t_e^* := \tau_{e_1}\tau_{e_2}\cdots\tau_{e_k}\tau_e\tau_{e_1}\tau_{e_2}\cdots\tau_{e_k}$ where $\{e_1,\dots,e_k\}$ are the elements that e covers.
- $\eta_e := t_{x_1}t_{x_2}\cdots t_{x_k}$ where $(x_1,x_2,\dots,x_k)$ is a linear extension of the subposet $\{x<e\}$ of P.
- $\tau_e^* := \eta_e t_e \eta_e^{-1}$

are defined as before, but now $t_e : OR(P) \to OR(P)$ and $\tau_e : \mathcal{C}(P) \to \mathcal{C}(P)$ are the *piecewise-linear* toggles.

$$\begin{array}{ccc} \mathcal{C}(P) & \xrightarrow{t_e^*} & \mathcal{C}(P) \\ \mathbf{OR}\downarrow & & \downarrow\mathbf{OR} \\ OR(P) & \xrightarrow[t_e]{} & OR(P) \end{array} \qquad \begin{array}{ccc} \mathcal{C}(P) & \xrightarrow{\tau_e} & \mathcal{C}(P) \\ \mathbf{OR}\downarrow & & \downarrow\mathbf{OR} \\ OR(P) & \xrightarrow[\tau_e^*]{} & OR(P) \end{array}$$

Future research

- By "detropicalizing" the operations, generalize toggles on chain polytopes to birational toggles, similar to what Einstein, Grinberg, Propp, Roby have done for toggles on order polytopes.
- How can we use the isomorphisms discussed here to translate homomesy results between antichains and order ideals, or between the chain and order polytopes?